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The homotopy Lie algebra of classifying spaces

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Abstract

Let X be a 1-connected CW-complex of finite type and L_X its rational homotopy Lie algebra. In this work, we show that there is a spectral sequence whose E^2 term is the Lie algebra $Ext_{UL_X}(Q, L_X)$, and which converges to the homotopy Lie algebra of the classifying space $B\,autX$. Moreover, some terms of this spectral sequence are related to derivations of L_X and to the Gottlieb group of X. © 1997 Elsevier Science B.V.

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0. Introduction

In this paper, X will denote a simply connected CW-complex of finite type. Fibrations whose fiber has the homotopy type of X are obtained, up to fiber homotopy equivalence, as pull back of the universal fibration $X \to B\,aut^{\bullet} X \to B\,autX$ [3,4]; here $autX$ denotes the monoid of self-homotopy equivalences of X, $aut^{\bullet}X$ the monoid of pointed self-homotopy equivalences of X, and B the Dold–Lashof classifying space of a monoid.

Denote by $\tilde{B}\,aut\,X$ the universal covering of $B\,aut\,X$. For the convenience of the reader, we recall the construction of models of $\tilde{B}autX$. For this, we shall use the theory of minimal models which is well developed in [14,9,2].

Denote by $(\Lambda Z, d)$ the Sullivan minimal model of X and by $(\mathbb{L}(V), \delta)$ its Quillen minimal model. We define from $(\Lambda Z, d)$ and $(\mathbb{L}(V), \delta)$ two Lie algebras of derivations. First, the differential Lie algebra $(Der\Lambda Z, D)$ is defined by [14]: in degree $k > 1$, we take the derivations of ΛZ decreasing degree by k. In degree one, we only consider

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the derivations θ which decrease degree by one and verify $[d, \theta] = 0$. The differential D is defined by $D\theta = [d, \theta] = d\theta - (-1)^{|\theta|}\theta d$.

In the same way, we define a differential Lie algebra $Der\mathbb{L}(V) = \bigoplus_{k\geq 1} Der_k(\mathbb{L}(V))$, where $Der_k(\mathbb{L}(V))$ is the vector space of derivations which increase the degree by k with the restriction that $Der_1(\mathbb{L}(V))$ is the vector space of derivations of degree one which commute with the differential δ.

Define the differential Lie algebra $(s\mathbb{L}(V) \underset{\sim}{\oplus} Der\mathbb{L}(V), \tilde{D})$ as follows:

- $s\mathbb{L}(V) \underset{\sim}{\oplus} Der\mathbb{L}(V)$ is isomorphic to $s\mathbb{L}(V) \oplus Der\mathbb{L}(V)$ as a graded vector space,
- If $\theta, \theta' \in Der\mathbb{L}(V); sx, sy \in s\mathbb{L}(V), [\theta, \theta'] = \theta\theta' - (-1)^{|\theta||\theta'|}\theta'\theta$, $[\theta, sx] = (-1)^{|\theta|}s\theta(x)$, $[sx, sy] = 0$,
- $\tilde{D}(\theta) = [\delta, \theta], \tilde{D}(sx) = -s\delta x + ad\,x$, where $ad\,x$ is the derivation of $\mathbb{L}(V)$ defined by $(ad\,x)(y) = [x, y]$.

Theorem [13–15]. *The differential Lie algebras* $(s\mathbb{L}(V) \underset{\sim}{\oplus} Der\mathbb{L}(V), \tilde{D})$ *and* $(Der\Lambda Z, D)$ *are models of the universal covering* $\tilde{B}\,aut\,X$ *of* $B\,aut\,X$.

In particular cases, there is a morphism between the two models above which induces an isomorphism in homology:

Theorem 1. *Let* $(\Lambda Z, d)$ *be the Sullivan minimal model of* X *and* $\mathbb{L}(W) = L_*(\Lambda Z, d)$, *there is a morphism*

$$\Psi : Der(\Lambda Z, d) \to (s\mathbb{L}(W) \underset{\sim}{\oplus} Der\mathbb{L}(W), \tilde{D}),$$

which induces an isomorphism in homology.

It is well known that a connected differential graded algebra $T(V)$, admits a $T(V)$ free acyclic differential module of the form $(T(V) \otimes (Q \oplus sV), D)$ [1, 10]. The differential is defined by

$$Dv = dv \otimes 1, \; Dsv = v \otimes 1 - S(\delta v \otimes 1), \tag{i}$$

where S is the Q-graded vector spaces map (of degree 1) defined by

$$\begin{aligned} &S(v \otimes 1) = 1 \otimes sv, \; S(1 \otimes (Q \oplus sV) = 0, \\ &S(ax) = (-1)^{|x|}a.S(x), \; \forall a \in TV, \, x \in TV \otimes (Q \oplus sV), \, |x| > 0. \end{aligned} \tag{ii}$$

The next result gives an interpretation of the rational homotopy Lie algebra of classifying spaces as a differential Ext [11].

Theorem 2. *Let* $(\mathbb{L}(V), \delta)$ *be a Quillen model of* X *and consider the Adams–Hilton acyclic construction* $P = (TV \otimes (Q \oplus sV), D)$. *There is an isomorphism of differential* *graded vector spaces*

$$\Phi : Hom_{TV}(P, \mathbb{L}(V)) \to (s\mathbb{L}(V) \underset{\sim}{\oplus} Der\mathbb{L}(V), \tilde{D}). \quad \text{(iii)}$$

Remark. If (L, δ) is a Quillen model of X, we deduce from [13] that $H^*_{Lie}(L, L)$ and $\pi_*(B\,aut\,X) \otimes Q$ are isomorphic as graded vector spaces. However, the theorem above asserts that $\pi_*(B\,aut\,X) \otimes Q \cong Ext_{TV}(Q, \mathbb{L}(V))$; therefore, since the differential Ext is invariant by quasi-isomorphism, we can compute $\pi_*(B\,aut\,X) \otimes Q$ using any differential graded Lie algebra model of X. In particular, if X is a coformal space, $\pi_*(B\,aut\,X) \otimes Q \cong Ext_{H_*(\Omega X, Q)}(Q, \pi_*(\Omega X) \otimes Q)$.

1. Proof of Theorem 2

Let $f \in Hom_{TV}(P, \mathbb{L}(V))$, define $\Phi(f) = (-1)^{|f|} sf(1) + \theta$, where θ is the derivation of $\mathbb{L}(V)$ of degree $|f| + 1$ verifying $\theta(v) = f(sv)$. It is clear that Φ is a one-one morphism of graded vector spaces. It remains to prove that Φ commutes with the differentials. On one hand,

$$\begin{aligned} D(\Phi(f)) &= D((-1)^{|f|} sx + \theta) \qquad \text{(iv)} \\ &= -(-1)^{|f|} s\delta x + (-1)^{|f|} ad\, x + [\delta, \theta], \quad \text{where } x = f(1). \end{aligned}$$

On the other hand,

$$\Phi(Df) = -(-1)^{|f|} s(Df)(1) + \theta', \quad \text{(v)}$$

where θ' is the derivation of $\mathbb{L}(V)$ defined $\theta'(v) = (Df)(sv)$. In order to verify the equalty of expressions (iv) and (v), observe first that

$$(Df)(1) = \delta f(1) = \delta x.$$

Next,

$$\begin{aligned} (Df)(sv) &= \delta(f(sv)) - (-1)^{|f|} f(d\, sv) \\ &= \delta\theta(v) - (-1)^{|f|} f(v \otimes 1 - S(dv \otimes 1)) \\ &= \delta\theta(v) - (-1)^{|f|} (-1)^{|f||v|} [v, f(1)] + (-1)^{|f|} f(S(dv \otimes 1)) \\ &= (-1)^{|f|} (ad\, x)(v) + \delta\theta(v) + (-1)^{|f|} \theta\delta(v) \\ &= (-1)^{|f|} (ad\, x)(v) + [\delta, \theta](v). \quad \square \end{aligned}$$

2. Proof of Theorem 1

Define a filtration $\mathscr{F}$ on the differential Lie algebra $Der(\Lambda Z, d)$ by the Lie differential subalgebras

$$F_p = \{\theta \in Der(\Lambda Z, d) \mid \theta(Z) \subset \Lambda^{\geq p} Z\},$$

This filtration is compatible with the differential, but we have $[F_p, F_q] \subset F_{p+q-1}$. To obtain a filtered Lie algebra in the usual sense, define $(s^{-1}F)_p = F_{p+1}$, so we have $[(s^{-1}F)_p, (s^{-1}F)_q] \subset (s^{-1}F)_{p+q}$. But, by commodity, we shall work with the filtration $\mathscr{F}$. This filtration determines a spectral sequence of differential graded Lie algebras (E^r, d^r) such that $E^2 \cong H_*(Der(\Lambda Z, d_2))$, where d_2 denotes the quadratic part of the differential d, and which converges to $H_*(Der(\Lambda Z, d))$.

Consider the differential Lie algebra $(\mathbb{L}(W), \delta_1 + \delta_2)$ obtained by applying the Quillen functor L_* to the commutative differential graded algebra $(\Lambda Z, d)$ [12]. Recall that $W \cong s^{-1}(\Lambda Z)^\vee$, is endowed with a filtration induced by the word length in ΛZ :

$$W_p = \bigoplus_{q \geq 1} W_{p,q}, \;\; W_{p,q} = s^{-1}((\Lambda^{p+1}Z)^{q+1})^\vee,$$

The differentials δ_1 and δ_2 are defined by

$$\begin{array}{l} \langle sa, \delta_1 s^{-1}b\rangle = (-1)^{|a|}\langle sda, s^{-1}b\rangle \\ \langle sa_1 \wedge sa_2; \delta_2 s^{-1}b\rangle = (-1)^{|a_2|}\langle sa_1a_2, s^{-1}b\rangle \\ a, a_1, a_2 \in \Lambda Z, b \in Hom(\Lambda Z, Q) \quad [12, 15], \end{array} \tag{vi}$$

and verify $\delta_1(W_{p,q}) \subset \bigoplus_{s<p} W_{s,q-1}$, $\delta_2(W_{p,q}) \subset (\mathbb{L}^2(W))_{p-1,q-1}$. First, filter P as follows: $P_0 = T(W) \otimes Q$, $P_1 = T(W) \otimes (Q \oplus (sW)_{1,*}) \cdots P_n = T(W) \otimes (Q \oplus (sW)_{\leq n,*})$.

A filtration of the complex $Hom_{T(W)}(P, M)$ is given by

$$F_n = \{f \in Hom_{T(W)}(P, L(W)) \mid f(P_{n-1}) = 0\}. \tag{vii}$$

This filtration determines a spectral sequence $(\bar{E}^r, \bar{d}^r)$, such that $\bar{E}^2 = Ext_{H_*(\Omega X, Q)}(Q, \pi_*(\Omega X) \otimes Q)$, which converges to $Ext_{T(W)}(Q, \mathbb{L}(W))$.

Define a morphism $\Psi : Der\Lambda Z \to s\mathbb{L}(W) \underset{\sim}{\oplus} Der\mathbb{L}(W) \cong Hom_{T(W)}(P, \mathbb{L}(W))$, as follows:

$\forall\, \theta \in Der\Lambda Z$, $\Psi(\theta) = su + \alpha$, where $u \in \mathbb{L}(W)$ and $\alpha \in Der\mathbb{L}(W)$ are defined by

$$\alpha(W) \subset W$$

and

$$\langle sx, \alpha(s^{-1}z^*)\rangle = \begin{cases} -(-1)^{|\theta||x|}\langle s\theta(x), s^{-1}z^*\rangle & \text{if } \theta(x) \in \Lambda^+ Z, \\ 0 & \text{otherwise,} \end{cases}$$

$$u \in W \text{ and } \langle sx, u\rangle = \begin{cases} \theta(x) & \text{if } \theta(x) \in Q, \\ 0 & \text{otherwise, } x, z \in \Lambda Z. \end{cases}$$

It is a straightforward computation to show that Ψ is a differential Lie algebra morphism which is compatible with filtrations. In particular,

$$Im\Psi = \{sx + \alpha \mid x \in W_0, \alpha(W_p) \subset \bigoplus_{q \leq p+1} W_q, adx + [\delta_2, \alpha] = 0\}. \tag{viii}$$

Therefore, by a spectral sequence argument [5], it remains to prove that the morphism $\Psi^2 : E^2 \to \bar{E}^2$ is an isomorphism. First, observe that $\bar{E}^2 = H_*(s\mathbb{L}(W) \underset{\sim}{\oplus} Der\,\mathbb{L}(W), \bar{D})$

where $(\mathbb{L}(W), \bar{\delta}) = L_*(\Lambda Z, d_2)$ and any cycle of $(s\mathbb{L}(W) \underset{\sim}{\oplus} Der\mathbb{L}(W), \bar{D})$ is homologue to a cycle in $Im\ \Psi$, therefore Ψ^2 is surjective.

On the other hand, let $sx+\alpha$ be a cycle in $Im\,\Psi$ which is a boundary in $(s\mathbb{L}(W) \underset{\sim}{\oplus} Der\ \mathbb{L}(W), \bar{D})$. Then, there is an element $sy + \beta \in s\mathbb{L}(W) \underset{\sim}{\oplus} Der\mathbb{L}(W)$ such that: $-s\delta' y + ad\ y + [\delta', \beta] = sx + \alpha$.

As $x \in W_0$, then $x = -\delta'(y) = 0$ and we can suppose that $y \in W_0$. Moreover, $\alpha(W_p) \subset \bigoplus_{q \leq p} W_q$, then $\alpha = \Psi(\theta)$ where θ is a derivation of ΛZ verifying $\theta(Z) \subset \Lambda^+ Z$. By induction on the bidegree of W and using specific properties of the bifiltered model [6], we define a derivation γ of $\mathbb{L}(W)$ such that $\gamma(W) \subset W$, $ad\ y + [\delta', \gamma] = \alpha$.

The derivation γ can be decomposed in a sum of two derivations γ_1 and γ_2 such that $\gamma_1(W_p) \subset \bigoplus_{q \leq p+1} W_q$, $\gamma_2(W_p) \subset \bigoplus_{q > p+1} W_q$. As the differential δ' is of bidegree $(-1, -1)$, then $[\delta', \gamma_2] = 0$, $ad\ y + [\delta', \gamma_1] = \alpha$, so we have the relations

$$[\delta'_1, \gamma_1] = \alpha,\ ad\ y + [\delta'_2, \gamma_1] = 0.$$

Therefore, according to (viii), $\alpha = D'(sy + \gamma_1)$ with $sy + \gamma_1 \in Im\ \Psi$, then α is a boundary in $Im\ \Psi$. □

Corollary 3. *There is a spectral sequence $E^r(X)$ of graded Lie algebras such that*

$$E^2(X) = Ext_{U\pi_*(\Omega X)\otimes Q}(Q,\ \pi_*(\Omega X) \otimes Q),$$

which converges to $\pi_*(\tilde{B}\ aut\ X) \otimes Q$.

We call this sequence the classifying space spectral sequence of X.

3. Properties of $\pi_*(B\ aut\ X) \otimes Q$

In this section, we describe some properties of classifying spaces using the classifying space spectral sequence.

3.1. The Gottlieb group

Recall that the Gottlieb group $G(X)$ of X is the image of the map induced in homotopy by the evaluation $autX \to X$ [8]; or equivalently, the image of the connecting map of the long exact sequence of homotopy groups induced by the universal fibration $X \to B\ aut^\bullet X \to B\ aut\ X$. In terms of Quillen models, we define an evaluation map $k : Hom_{T(W)}(P,\ \mathbb{L}(W)) \to \mathbb{L}(W)$ by $k(f) = f(1)$, and consider the induced map in homology $H_*(k) : Ext_{T(W)}(Q,\ \mathbb{L}(W)) \to H_*(\mathbb{L}(W),\ \delta)$.

It is clear that $E^2_{0,*}(X) \cong Hom_{UL_X}(Q,\ L_X)$ is isomorphic to the center of the Lie algebra L_X. The result above gives more precision about the relation between the classifying space spectral sequence and the Gottlieb group on one hand, and the relation between the morphism k and the topological evaluation map on the other hand.

Proposition 4. (a) $H_*(k) = \pi_*\Omega(ev) \otimes Q$,
(b) $E^{\infty}_{0,*}(X) = G(X_0) = Im\, H_*(k)$.

Remark. If X is a coformal space, $E^{\infty}_{0,*}(X) = E^{2}_{0,*}(X)$; then the Gottlieb group of X_0 coincides with the center of $\pi_*(\Omega X) \otimes Q$.

Proof. We have a commutative diagram

$$\begin{array}{ccc} Ext_{T(W)}(Q,\, \mathbb{L}(W)) & \xrightarrow{\cong} & H_*(s\mathbb{L}(W) \underset{\sim}{\oplus} Der\mathbb{L}(W)) \\ & {\scriptstyle H_*(k)}\searrow \quad \swarrow{\scriptstyle h} & \\ & H_*(\mathbb{L}(W), \delta) & \end{array}$$

where h is defined by $h([sx + \theta]) = [(-1)^{|x|}x]$. Then, assertion (a) is a direct consequence of the fact that h is the connecting morphism of the homotopy exact sequence of the universal fibration $X \to B\ aut^{\bullet}X \to B\ autX$ [15]. From the classifying space spectral sequence, we have the relation $E^{\infty}_{0,*}(X) = Im\,(H_*(k))$.

Recall that the Gottlieb group of X_0 is isomorphic to the kernel of the map induced in homology by the adjunction map

$$ad : (\mathbb{L}(W), \delta) \to Der\,(\mathbb{L}(W), \delta) \quad [15].$$

To prove the assertion (b), it is sufficient to show that $Im\, H_*(k) = Ker(H_*(ad))$.

First, we show that $Im\, H_*(k) \subset Ker(H_*(ad))$. Let $[x] \in H_*(\mathbb{L}(W),\, \delta)$ such that $[x] = H_*(k)([f]) = [f(1)]$. Therefore, $ad\, x = [\delta,\, \theta]$, where θ is the derivation of $\mathbb{L}(W)$ defined by $\theta(y) = f(sy)$, $\forall\, y \in W$, then $[x] \in Ker\, H_*(ad)$.

On the other hand, if $[x] \in Ker\, H_*(ad)$, there is a derivation θ of $\mathbb{L}(W)$ such that $ad\, x = [\delta,\, \theta]$ and in this case $[x] = (H_*(k))([f])$ where f is defined by $f(1) = x$ and $f(sy) = \theta(y)$. □

We deduce from the classifying space spectral sequence that the homology class of $x \in \mathbb{L}(W)$ represents a Gottlieb element if only if the $T(W)$-modules morphism $f : T(W) \otimes Q \to \mathbb{L}(W)$ defined by $f(1) = x$ can be extended to a morphism $\bar{f}$ of $T(W)$-modules on $T(W) \otimes (Q \oplus sW)$ such that $D\bar{f} = 0$. This is the analogue of the description of Gottlieb elements given in ([7]).

3.2. Finite CW complexes

The next proposition describes an important property of the classifying space spectral sequence of a finite CW-complex.

Proposition 5. *Let X be a finite simply connected CW-complex, there is an integer n such that $E^{\infty}_{q,*}(X) = 0$ for all q greater than n.*

Proof. Define $n = sup\{k, H^k(X, Q) \neq 0\}$. Let $\theta \in F_p, p > n$ such that $[d, \theta] = 0$, we shall prove that θ is a boundary i.e. there is a derivation θ' such that $\theta = [d, \theta']$. Suppose that θ' is defined on $Z^{<r}$ such that $\theta = [d, \theta']$ on $Z^{<r}$. Let $x \in Z^r$, compute

$$\begin{aligned} d(\theta(x) + (-1)^{|\theta|}\theta'(dx)) &= d\theta(x) + (-1)^{|\theta|}d\theta'(dx) \\ &= (-1)^{|\theta|}\theta(dx) + (-1)^{|\theta|}d\theta'(dx) \\ &= (-1)^{|\theta|}(\theta - [d,\theta])(dx) \\ &= 0. \end{aligned}$$

Therefore there is an element y in ΛZ such that $\theta(x) + (-1)^{|\theta|}\theta'(dx) = dy$. Then define $\theta'(x) = y$, then θ and $[d, \theta']$ agree on $Z^{\leq r}$. □

Corollary 6. *Let X be a finite simply connected CW-complex such that $G(X_0) = 0$, then $E^{\infty}_{\geq 2}$ is a nilpotent ideal of $\pi_*(\Omega\tilde{B}\, aut\, X) \otimes Q$.*

Proposition 7. *Let X be finite simply connected CW-complex such that $G(X_0) \neq 0$. If $E^{\infty}_{\geq 2}(X) = 0$, then there is a non trivial element in the center of $\pi_*(\Omega\tilde{B}\, aut\, X) \otimes Q$.*

Proof. Denote by $(\Lambda Z, d)$ the Sullivan minimal model of X. Since the rational LS-category of X is finite, the Gottlieb group of X_0 is of finite dimensional [7]. Let $\alpha \in G(X_0) \cong E^{\infty}_0(X)$ be an element of the highest degree, say n, and $\beta \in \pi_*(\Omega\tilde{B} aut X) \otimes Q$. As $E^{\infty}_{\geq 2}(X) = 0$, $\beta = \beta_0 + \beta_1$ where $\beta_i \in E^{\infty}_i(X)$. Therefore, $[\alpha, \beta] = [\alpha, \beta_1] + [\alpha, \beta_0] = [\alpha, \beta_1]$. Since $[\alpha, \beta_1] \in E^{\infty}_0(X)$, $[\alpha, \beta_1]$ corresponds to a Gottlieb element of X_0 of degree greater than n, then $[\alpha, \beta_1] = 0$, so α is in the center of $\pi_*(\Omega\tilde{B}\, aut\, X) \otimes Q$. □

3.3. *Derivations of $\pi_*(\Omega X) \otimes Q$*

We cannot expect to recover the homotopy groups of $B\, aut\, X$ from derivations of $\pi_*(\Omega X) \otimes Q$, even if X is a coformal space. Since $[E^{\infty}_1, E^{\infty}_1] \subset E^{\infty}_1$, then E^{∞}_1 is endowed with a Lie algebra structure inherited from E^2_1. As we shall see in the next result, E^2_1 is related, in a certain manner, to derivations of $\pi_*(\Omega X) \otimes Q$.

Proposition 8. *There is a Lie algebra isomorphism*

$$Der(\pi_*(\Omega X) \otimes Q)/ad\, L \to E^2_{1,*},$$

where $ad\, L$ is the ideal of $Der(\pi_*(\Omega X) \otimes Q)$ generated by inner derivations.

Remark. If X is a coformal space, then $Der(\pi_*(\Omega X) \otimes Q)/ad\, L$ is a sub Lie algebra of $\pi_*(B\, aut\, X) \otimes Q$.

Proof. Recall that $C^*(\pi_*(\Omega X) \otimes Q) \cong (\Lambda Z, d_2)$, where $Z^n = s^{-1}Hom(\pi_{n-1}(\Omega X) \otimes Q, Q)$. Define $\phi : Der(\pi_*(\Omega X) \otimes Q) \to E^2_{1,*}$ as follows. Let θ be a derivation of

$\pi_*(\Omega X) \otimes Q$, define a map $\phi_\theta : Z \to Z$ by $< \phi_\theta(s^{-1}z), sx >= -(-1)^{|\theta||sx|} < s^{-1}z, s\theta(x) >$, and extend it as a derivation on ΛZ. A straigthforward computation shows that ϕ is compatible with the Lie bracket; moreover, $[d_2, \phi_\theta] = 0$ since θ is a Lie algebra derivation. Therefore, ϕ_θ represents an element of $E^2_{1,*}$.

Let α be a derivation of ΛZ such that $\alpha(Z) \subset \Lambda^+ Z$ and $[d_2, \alpha] = 0$. Write $\alpha = \alpha_1 + \cdots + \alpha_i \cdots$ where $\alpha_i(Z) \subset \Lambda^i Z$. As $[d_2, \alpha_1] = 0$, there is a derivation θ of $\pi_*(\Omega X) \otimes Q$ such that $\phi_\theta = \alpha_1$. This shows that the map ϕ is surjective.

It remains to show that $Ker\, \phi \cong ad\, L$, where $ad\, L$ is the ideal of inner derivations of $\pi_*(\Omega X) \otimes Q$.

Let θ be an inner derivation of $\pi_*(\Omega X) \otimes Q$, there is an element a in $\pi_*(\Omega X) \otimes Q$ such that $\theta(x) = [a, x]$, $\forall x$. Consider the element $z = s^{-1}a^* \in Z$ and define a map α on Z such that $\alpha(z) = 1$ and extend it to a derivation α such that $\phi_\theta - [d_2, \alpha] = 0$ in $E^2_{1,*}$, therefore $ad\, L \subset Ker\, \phi$. Inversely if $\phi_\theta - [d_2, \alpha] = 0$ in $E^2_{1,*}$, $\alpha = \alpha_0 + \cdots \alpha_i + \cdots$ with $\alpha_i(Z) \subset \Lambda^i Z$, then $\phi_\theta = [d_2, \alpha_0]$; therefore, the dual of α_0 is an inner derivation which coincides with θ. □

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References

[1] J.F. Adams and P.J. Hilton, On the chain algebra of a loop space, Comment. Math. Helvetici 30 (1956) 305–330.

[2] H.J. Baues and J.M. Lemaire, Minimal models in homotopy theory, Math. Ann. 225 (1977) 219–242.

[3] A. Dold, Halbexacte Homotopiefunctoren, Lecture Notes in Mathematics, Vol. 12 (Springer, Berlin, 1966).

[4] E. Dror and A. Zabrodsky, Unipotency and nilpotency in homotopy equivalences, Topology 18 (1979) 187–197.

[5] S. Eilenberg and J.C. Moore, Limits and spectral sequences, Topology 1 (1962) 1–23.

[6] Y. Félix, Modèles bifiltrés: une plaque tournante en homotopie rationnelle, Can. J. Math. 23 (1981) 1448–1458.

[7] Y. Félix and S. Halperin, Rational L.S.-category and its application, Trans. Amer. Math. Soc. 273 (1982) 1–37.

[8] D.H. Gottlieb, Evaluation subgroups of homotopy groups, Amer. J. Math. 91 (1969) 729–756.

[9] S. Halperin, Lectures on Minimal Models, Mémoire Société Mathématique de France, Vols. 9–10 (1983).

[10] J.M. Lemaire, Algèbres connexes et homologie des espaces de lacets, Lectures Notes in Mathematics, Vol. 422 (Springer, Berlin, 1974).

[11] J.C. Moore, Algèbre homologique et homologie des espaces classifiants, Séminaire H. Cartan (1959/1960), exposé no 7.

[12] D. Quillen, Rational homotopy theory, Ann. Math. (2) 90 (1969) 205–295.
[13] M. Schlessinger and J. Stasheff, Deformations theory and rational homotopy type, Publ. IHES, to appear.
[14] D. Sullivan, Infinitesimal computations in topology, Publ. IHES 47 (1977) 269–331.
[15] D. Tanré, Homotopie rationnelle; modèles de Chen, Quillen, Sullivan, Lecture Notes in Mathematics, Vol. 1025 (Springer, Berlin, 1983).